Univerzitet u Zenici
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Odsjek: Matematika i informatika
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Pismeni ispit iz Linearne algebre

Pravila: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Zadatak prepisati sa table.

2. Neka je $Q : \mathcal{P}_3 \to \mathcal{P}_3$ ($\mathcal{P}_3$ označava prostor svih polinoma stepena ≤ 3) linearni operator dat sa

$$Q(p) = \text{polinom stepena 2 čiji graf prolazi tačkama}(-1; p(-1)),\ (0; p(0)) \text{ i } (1; p(1)).$$

(a) Odrediti matricu operatora Q (matricu koordinata) u odnosu na standardnu bazu.
(b) Odrediti (direktni) komplement prostora $\ker(Q)$ (koji nije ortogonalni komplement).

3. Zadan je linearni operator $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathcal{P}_3$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (3a + 2c - d)x^3 + (3b - c - d)x^2 + (2a + b + c - d)x + (a + 2b - d)$$

Odrediti ortonormiranu bazu za $\mathrm{im}(T)$ (Koristiti standardni unutrašnji proizvod u $\mathcal{P}_3$ definisan sa $\langle a_3x^3 + a_2x^2 + a_1x + a_0, b_3x^3 + b_2x^2 + b_1x + b_0\rangle = a_3b_3 + a_2b_2 + a_1b_1 + a_0b_0$).

4. U unitarnom prostoru $\mathbb{R}^4$, sa skalarnim proizvodom

$$\langle x, y\rangle = x_1y_1 + 2x_2y_2 + x_3y_3 + 2x_4y_4$$

zadan je podprostor $\mathcal{V}$ razapet (generisan) vektorima $v_1 = (1, 0, 1, 0)^\top$ i $v_2 = (1, 0, 1, 1)^\top$. Prikažite vektor $x = (4, 2, 2, 4)^\top$ u obliku $x = v + w$, gdje je $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

(#) Zadan je skup

$$V=\left\{\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4\end{bmatrix}\in Mat_{2\times2}(\mathbb{C}) : z_1-2\overline{z_2}+z_3=0,\ z_1+\overline{z_2+z_3}+z_4=0\right\}.$$

Dokazati da je V realni vektorski podprostor prostora $Mat_{2\times2}(\mathbb{C})$ te mu nađite neku bazu i odredite dimenziju.

Rj.

Prisjetimo se:

Neprazan skup S vektorskog prostora M je podprostor od M akko vrijedi

(A1) $x,y\in S \Rightarrow x+y\in S$

(M1) $x\in S \Rightarrow \alpha x\in S$ za $\forall\alpha\in\mathbb{R}$

(A1) $M,N\in V \Rightarrow M+N\in V$

$$M=\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4\end{bmatrix} \quad m_1-2\overline{m_2}+m_3=0 \quad ; \quad m_1+\overline{m_2+m_3}+m_4=0$$

$$N=\begin{bmatrix} n_1 & n_2 \\ n_3 & n_4\end{bmatrix} \quad n_1-2\overline{n_2}+n_3=0 \quad ; \quad n_1+\overline{n_2+n_3}+n_4=0$$

$$M+N=\begin{bmatrix} m_1+n_1 & m_2+n_2 \\ m_3+n_3 & m_4+n_4\end{bmatrix} \quad \text{i vrijedi} \quad (m_1+n_1)-2\underbrace{(\overline{m_2}+\overline{n_2})}_{=\overline{m_2+n_2}}+(m_3+n_3)=0$$

$$(m_1+n_1)+(\overline{m_2+n_2)+(m_3+n_3})+(m_4+n_4)=0 \quad \Rightarrow \quad M+N\in V$$

(M1) $\alpha\in\mathbb{R},\ Z\in V \Rightarrow \alpha Z\in V$

$$Z=\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4\end{bmatrix} \quad z_1-2\overline{z_2}+z_3=0, \quad z_1+\overline{z_2+z_3}+z_4=0$$

$$\alpha Z_1 = \begin{bmatrix} \alpha z_1 & \alpha z_2 \\ \alpha z_3 & \alpha z_4 \end{bmatrix}, \qquad \alpha z_1 - 2\alpha\overline{z_2} + \alpha z_3 = \alpha\underbrace{(z_1 - 2\overline{z_2} + z_3)}_{=0} = 0$$

$$\alpha z_1 + \overline{\alpha z_2 + \alpha z_3} + \alpha z_4 = \alpha\underbrace{(z_1 + \overline{z_2 + z_3} + z_4)}_{=0} = 0$$

$$\Rightarrow \alpha Z_1 \in Mat_{2\times 2}(\mathbb{C})$$

Pa prema (A1) i (M1) V jest realni vektorski podprostor.

Da bismo odredili bazu i dimenziju unutar prostora V posmatrajmo prostor V' definisan sa

$$V' = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; z_1 - 2\overline{z_2} + z_3 = 0,\ z_1 + \overline{z_2 + z_3} + z_4 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; a_1 + ib_1 - 2\overline{(a_2 + ib_2)} + (a_3 + ib_3) = 0,\ a_1 + ib_1 + \overline{(a_2 + ib_2) + (a_3 + ib_3)} + a_4 + ib_4 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \\ a_4 + ib_4 \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; a_1 - 2a_2 + a_3 = 0,\ b_1 + 2b_2 + b_3 = 0,\ a_1 + a_2 + a_3 + a_4 = 0,\ b_1 - b_2 - b_3 + b_4 = 0 \right\}$$

$$= \left\{ \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \\ a_4 + ib_4 \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = 0 \wedge \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0 \right\}$$

Rješimo dva dobijena sistema i time odredimo koeficijente za a i b.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \sim \ldots \sim \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \end{array}\right] \end{array} \Rightarrow$$

dvije promjenjive uzimamo proizvoljno $a_3 = t$, $a_4 = 3s$, $t, s \in \mathbb{R}$

$$a_1 = -a_3 - \frac{2}{3}a_4 = -t - 2s$$

$$a_2 = -\frac{1}{3}a_4 = -s$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -t-2s \\ -s \\ t \\ 3s \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 0 \\ 3 \end{pmatrix} s$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right] \sim \ldots \sim \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{array}\right] \end{array} \Rightarrow$$

dvije promjenjive uzimamo proizvoljno $b_3 = 3u$, $b_4 = 3v$, $u, v \in \mathbb{R}$

$$b_1 = \frac{1}{3}b_3 - \frac{2}{3}b_4 = u - 2v$$

$$b_2 = -\frac{2}{3}b_3 + \frac{1}{3}b_4 = -2u + v$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} u-2v \\ -2u+v \\ 3u \\ 3v \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} u + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} v$$

$$V' = \left\{ \begin{pmatrix} a_1+ib_1 \\ a_2+ib_2 \\ a_3+ib_3 \\ a_4+ib_4 \end{pmatrix} \in \mathbb{C}^4 \;\middle|\; \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 0 \\ 3 \end{pmatrix} s,\; t, s \in \mathbb{R}, \right.$$

$$\left. \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} u + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} v,\; u, v \in \mathbb{R} \right\},$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix} i, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} i \right\}.$$

Time:

Dimenzija prostora V je 4 a jedna od baza je

$$\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} i & -2i \\ 3i & 0 \end{bmatrix}, \begin{bmatrix} -2i & i \\ 0 & 3i \end{bmatrix} \right\}$$

(#) Neka je $Q: \mathcal{P}_3 \to \mathcal{P}_3$ ($\mathcal{P}_3$ označava prostor polinoma stepena ≤ 3) linearni operator zadan sa

$Q(p) =$ polinom stepena 2 čiji graf prolazi tačkama $(-1; p(-1))$, $(0; p(0))$ i $(1; p(1))$.

(a) Odrediti matricu operatora Q (matricu koordinata) u odnosu na standardnu bazu.

(b) Odrediti (direktni) komplement prostora $\ker(Q)$ (koji nije ortogonalni komplement).

Rj.

$$p(x) = ax^3 + bx^2 + cx + d$$

$$p(-1) = -a + b - c + d$$

$$p(0) = d$$

$$p(1) = a + b + c + d$$

Prvo odredimo djelovanje operatora Q.

Trebamo odrediti koeficijente polinoma $g(x)$ stepena 2 čiji graf prolazi tačkama $(-1; -a+b-c+d)$, $(0; -a+b-c+d)$ i $(1; a+b+c+d)$.

Označimo sa $r_2x^2 + r_1x + r_0$ polinom $g(x)$ tj. $g(x) = r_2x^2 + r_1x + r_0$.

Tada je

$$g(-1) = r_2 - r_1 + r_0$$

$$g(0) = r_0 \quad \Rightarrow \quad r_0 = d$$

$$g(1) = r_2 + r_1 + r_0$$

$$g(-1) = r_2 - r_1 + d = -a + b - c + d$$

$$g(1) = r_2 + r_1 + d = a + b + c + d$$

Sad imamo

$$r_2 - r_1 = -a + b - c \quad (1)$$

$$r_2 + r_1 = a + b + c \quad (2)$$

$(1)+(2)$: $\quad 2r_2 = 2b \Rightarrow r_2 = b \quad \overset{(2)}{\Rightarrow} \quad r_1 = a + c$ Time smo dobili

$$Q(ax^3+bx^2+cx+d) = bx^2+(a+c)x+d$$

(a)

Da bismo olakšali rješenje zadatka umjesto prostora $\mathcal{P}_3$ posmatraj prostor $\mathbb{R}^4$ i linearni operator $Q': \mathbb{R}^4 \to \mathbb{R}^3$ definisan na sljedeći način

$$Q'\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ a+c \\ d \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Time smo dobili da je za bazu $\mathcal{G} = \{x^3, x^2, x, 1\}$

$$[Q]_{\mathcal{G}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $\ker(Q') = \{x \mid Q'x = 0\}$

$x_1 + x_3 = 0$
$x_2 = 0$
$x_4 = 0$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{I_v \leftrightarrow II_v}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow$$ 1 promjenjivu uzimamo proizvoljno npr. $x_3 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s, \ s \in \mathbb{R} \quad \left(= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} s, \ s \in \mathbb{R} \right)$$

$$\ker(Q) = \{ax^3 - ax \mid a \in \mathbb{R}\}$$

Označimo sa $\mathcal{M} = \ker(Q)$. Trebamo odrediti $\mathcal{N}$ tako da

$$\mathcal{M} \oplus \mathcal{N} = \mathcal{P}_3 \iff \begin{array}{l} \mathcal{M} \cap \mathcal{N} = \{0\} \\ \mathcal{M} + \mathcal{N} = \mathcal{P}_3 \end{array}$$

Prisjetimo se

Ako su $\mathcal{B}_X$ i $\mathcal{B}_Y$ baze za X i Y (podprostora V) tada

$V = X \oplus Y \iff \forall v \in V\ \exists!\, x \in X, y \in Y$ t.d. $v = x + y$ $\iff \mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$ i $\mathcal{B}_X \cup \mathcal{B}_Y$ je baza za V

Znamo da osnovne kolone u A generišu $\mathrm{im}(A)$.

$$\ker(Q') = \mathrm{im}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right)$$

$$\mathbb{R}^4 = \mathrm{im}\left(\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pa ako je $M' = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}\right\}$ tada je $N' = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

iz čega slijedi da je (direktni) komplement za $\ker(Q)$ $(=M)$

$$N = \{ ax^3 + bx^2 + c \mid a, b, c \in \mathbb{R} \}$$

(#) Zadan je linearni operator $T: Mat_{2\times 2}(\mathbb{R}) \longrightarrow \mathcal{P}_3$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (3a+2c-d)x^3 + (3b-c-d)x^2 + (2a+b+c-d)x + (a+2b-d).$$

Odrediti ortonormiranu bazu za $im(T)$. (Koristiti standardni unutrašnji proizvod u $\mathcal{P}_3$ definisan sa

$$\langle a_3x^3+a_2x^2+a_1x+a_0,\ b_3x^3+b_2x^2+b_1x+b_0 \rangle = a_3b_3+a_2b_2+a_1b_1+a_0b_0).$$

Rj. Da bismo odredili ortonormiranu bazu za $im(T)$ prvo trebamo odrediti bilo kakvu bazu, a da bismo sebi olakšali posao umjesto $Mat_{2\times 2}(\mathbb{R})$ i $\mathcal{P}_3$ posmatrajmo prostor $\mathbb{R}^4$ i operator T' definisan sa

$$T' : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longrightarrow \begin{pmatrix} 3a+2c-d \\ 3b-c-d \\ 2a+b+c-d \\ a+2b-d \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 & 2 & -1 \\ 0 & 3 & -1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} - im(A)$$

Prisjetimo se

<u>Osnovne kolone u A generišu $im(A)$.</u>

$$\begin{pmatrix} 3 & 0 & 2 & -1 \\ 0 & 3 & -1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & 0 & -1 \end{pmatrix} \sim \dots \sim \begin{pmatrix} \boxed{1} & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \boxed{1} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Time možemo zaključiti da je

$$im(T) = span\{3x^3+2x+1,\ 3x^2+x+2\}$$

Zanimljivo pitanje: Kako provjeriti da je ovaj rezultat za $im(T)$ tačan?

Da bismo odredili ortonormiranu bazu primjenimo Gram-Schmidtovu proceduru.

Prisjetimo se (jednog od načina):

— polazimo od baze $\{u_1, u_2, u_3, \dots\}$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{\langle v_1, u_3 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, u_3 \rangle}{\|v_2\|^2} v_2$$

$$\vdots$$

ovim postupkom prvo bi dobili ortogonalne vektore $\{v_1, v_2, v_3, \dots\}$.

Pa krenimo redom

$$v_1 = 3x^3 + 2x + 1$$

$u_2 = 3x^2 + x + 2$, $\quad \langle v_1, u_2 \rangle = 0 + 0 + 2 + 2 = 4$

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = 9 + 4 + 1 = 14$$

$$v_2 = 3x^2 + x + 2 - \frac{\cancel{4}^{\,2}}{\cancel{14}_{\,7}}(3x^3 + 2x + 1) = -\frac{6}{7}x^3 + 3x^2 + \frac{3}{7}x + \frac{12}{7}$$

$\frac{28}{14}$

Ortogonalna baza za $im(T)$ je $\left\{3x^3 + 2x + 1, -\frac{6}{7}x^3 + 3x^2 + \frac{3}{7}x + \frac{12}{7}\right\}$

a kako je $\|3x^3 + 2x + 1\|^2 = 14$ i $\|-\frac{6}{7}x^3 + 3x^2 + \frac{3}{7}x + \frac{12}{7}\|^2 = \frac{90}{7}$

to je ortonormirana baza za $im(T)$

$$\left\{\frac{3}{\sqrt{14}}x^3 + \frac{2}{\sqrt{14}}x + \frac{1}{\sqrt{14}},\ -\frac{2\sqrt{7}}{7\sqrt{10}}x^3 + \frac{\sqrt{7}}{\sqrt{10}}x^2 + \frac{\sqrt{7}}{7\sqrt{10}}x + \frac{4\sqrt{7}}{7\sqrt{10}}\right\}$$

$\frac{\sqrt{7}}{3\sqrt{10}}$

(#) U unitarnom prostoru $\mathbb{R}^4$, sa skalarnim proizvodom

$$\langle x, y\rangle = x_1y_1 + 2x_2y_2 + x_3y_3 + 2x_4y_4$$

zadan je podprostor $\mathcal{V}$ razapet vektorima $v_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ i $v_2 = (1,0,1,1)^T$. Prikažite vektor $x = (4,2,2,4)^T$ u obliku $x = v + w$, gdje je $v \in \mathcal{V}$, $w \in \mathcal{V}^\perp$.

Rj.

Prisjetimo se

Ortogonalni komplement

Za podskup M unitarnog prostora $\mathcal{V}$, ortogonalni komplement $M^\perp$ od M je definisan sa

$$M^\perp = \{x \in \mathcal{V} \mid \langle m, x\rangle = 0 \text{ za } \forall m \in M\}$$

U našem slučaju primjetimo da je

$$\mathcal{V} = \text{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}\right\}$$

Kako je $\dim \mathbb{R}^4 = 4$, $\dim \mathcal{V} = 2$ i $\mathbb{R}^4 = \mathcal{V} \oplus \mathcal{V}^\perp$ to je $\dim \mathcal{V}^\perp = 2$. Odredimo vektore $v_3 = (a,b,c,d)$ i $v_4 = (e,f,g,h)$ takve da

$$\langle v_1, v_3\rangle = 0 \Rightarrow a + c = 0$$
$$\langle v_2, v_3\rangle = 0 \Rightarrow a + c + 2d = 0$$
$$\langle v_1, v_4\rangle = 0 \Rightarrow e + g = 0$$
$$\langle v_2, v_4\rangle = 0 \Rightarrow e + g + 2h = 0$$

$$\begin{aligned} a + c &= 0 \\ a + c + 2d &= 0 \end{aligned}$$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right] \end{array} \overset{II_v - I_v}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{c} a = -c \\ d = 0 \end{array}$$

$$\left.\begin{array}{r} rang(A)=rang(\bar{A})=2 \\ broj\ nepoznatih=4 \end{array}\right\} \Rightarrow$$ dvije promjenjive uzimamo proizvoljno npr. $b=t$, $c=s$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -s \\ t \\ s \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} t$$

Primjetimo da smo u procesu određivanja vektora v_3 u stvari odredili i vektor v_4. Time smo dobili:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Lagana provjera nam pokazuje da je $v_1 \perp v_3$, $v_1 \perp v_4$, $v_2 \perp v_3$, $v_2 \perp v_4$.

Da li je skup $\{v_1, v_2, v_3, v_4\}$ linearno nezavisan?

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underset{III_v - I_v}{\overset{II_v \leftrightarrow IV_v}{\sim}} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ skup $\{v_1, v_2, v_3, v_4\}$ jest linearno nezavisan.

Ostalo je još da razložimo vektor x preko vektora v_1, v_2, v_3, v_4. Odredimo α, β, γ i δ t.d.

$$\begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} \alpha+\beta-\gamma=4 \\ \alpha+\beta+\gamma=2 \end{array}\right\} \overset{+}{\Rightarrow} 2\alpha+2\beta=6$$

$\delta=2$, $\beta=4$ $\qquad 2\alpha+8=6$

$\alpha=-1$

$$\Rightarrow \alpha=-1, \ \beta=4, \ \gamma=-1, \ \delta=2$$

Možemo zaključiti da je

$$\underbrace{\begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 0 \\ 3 \\ 4 \end{pmatrix}}_{v} + \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}}_{w}$$ gdje su $v = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 4 \end{pmatrix} \in V$ i $w = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \in V^{\perp}$